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# Super twistor space and $N = 2$ supersymmetric instantons

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**Abstract**

A relationship between $N = 2$ supersymmetric Yang–Mills instantons on the Euclidean four space and certain holomorphic super vector bundles over super twistor space is investigated. We give the ADHM-matrix solutions of $N = 2$ supersymmetric instantons.


*MSC:* Differential geometry

*PSC:* High energy physics-theory

*JGP SC:* Differential geometry

*Keywords:* Supersymmetry; Super twistor; Instantons; ADHM-matrix

## 1. Introduction

In 1989, Harnad et al. [9] showed that there is a one-to-one correspondence between $N$-supersymmetric Yang–Mills fields over the complex super vector space $C^{4|4N}$ and certain holomorphic super vector bundles over the space of super null lines $L^{5|2N}$. Theorem 2.1 in Section 2 has a long history. Using Ferber's [7] extension of twistor theory to complex super vector space, Witten [19] gave a proof for $N = 0$ and 3 using supersymmetry. Independently, Green and coworkers [12] gave a proof for $N = 0$. Manin [14] reformulated super twistor theory using super flag manifolds and outlined a cohomological proof for $0 \leq N \leq 3$ which generalized the earlier cohomological proof for the case $N = 0$ [11].

On the other hand, the ADHM description [2,4] was first discovered using twistor methods which go back to Ward [17]. The twistor space of four sphere $S^4$ is complex projective three

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space $P^3$ and Ward showed that there is a one-to-one correspondence between instantons over open sets in $S^4$ and certain holomorphic vector bundles over the corresponding open sets in $P^3$, it so called *Penrose–Ward correspondence*. So the problem of describing all instantons is reduced to the description of holomorphic vector bundles over $P^3$. In this framework the solutions are obtained from monads over $P^3$, and then the Penrose–Ward correspondence is used to pass back to $S^4$. See Atiyah [1] and Atiyah et al. [2–4] for this part of the theory.

The purpose of this paper is to present a $N = 2$ supersymmetric generalized instantons of some of the above-mentioned results. The organization of the paper is as follows. Section 2 describes a review of some known facts about supermanifold, spinors, $N = 2$ supersymmetry and super connection, etc. In Section 3 we define the super twistor space, self-dual super plane and double fibration of super twistor diagram. In Section 4 we define the $N = 2$ supersymmetric instantons. We also show the equivalent conditions for $N = 2$ supersymmetric instantons. In Section 5 we generalize the Penrose–Ward correspondence $N = 2$ supersymmetric instantons in the case of complex super vector space $C^{4|8}$ with structure group $GL(p|q; C)$. The idea of proof is to use the self-dual super plane. We also present to describe the Penrose–Ward correspondence which is restricting to the real super vector space $R^{4|8}$ with structure group $GL(n; C)$ and $SU(n)$. In Section 6 we give the ADHM-matrix solutions of $N = 2$ supersymmetric instantons.

## 2. Preliminaries

Let $E$ be a rank-$n$ vector bundle over a $m$-dimensional manifold $M$. Then let $\wedge E$ be the exterior algebra of $E$, and let $\mathcal{O}(\wedge E)$ be the locally free sheaf of sections of $\wedge E$ [13,14].

**Definition 2.1.** The ringed space $(M, \mathcal{O}(\wedge E))$ is called a split supermanifold of dimension $m|n$.

Let $C^m$ be a $m$-dimensional complex vector space. The typical example is the complex (or real) super vector space $C^{m|n}$ (or $R^{m|n}$). Let $E$ be the trivial $C^n$-bundle over $C^m$. Then

$$\begin{aligned}\bigwedge E &= \bigwedge(C^m \times C^n) = S(C^m)^* \otimes \bigwedge(C^n)^* = C[x^1, x^2, \dots, x^m] \otimes \bigwedge(C^n)^* \\ &= C^\infty(C^m) \otimes \bigwedge(C^n)^*.\end{aligned}$$

Hence the ringed space $(C^m, \mathcal{O}(C^\infty(C^m) \otimes (C^n)^*))$ is called the complex super vector space and is denoted by $C^{m|n}$. Consider

$$\mathcal{S} = (C^2, \epsilon),$$

where $\epsilon$ is a skew-symmetric nondegenerate complex bilinear form. Since $\epsilon$ is nondegenerate, we can use $\epsilon$ to identify $\mathcal{S}$ with $\mathcal{S}^*$, the complex-linear dual of $\mathcal{S}$. Then we have

$$\begin{aligned}&\mathcal{S} \otimes_R C = \mathcal{S}^+ \oplus \mathcal{S}^-, \qquad T_x R^4 \otimes_R C \cong \mathcal{S}^+ \otimes \mathcal{S}^-, \\ &\mathcal{S}^* \otimes_R C = \mathcal{S}_+ \oplus \mathcal{S}_-, \qquad T_x^* R^4 \otimes_R C \cong \mathcal{S}_+ \otimes \mathcal{S}_-,\end{aligned}$$

where $\mathcal{S}^+$ and $\mathcal{S}^-$ are the $i$ and $-i$ eigenspaces, respectively, of the almost complex structure of $S$ extended in a $C$-linear fashion to $\mathcal{S} \otimes_R C$. We treat $\mathcal{S}^* \otimes_R C$ in the same fashion. The mapping

$$(\epsilon^{AB}) : \mathcal{S}_+ \to \mathcal{S}^+, \qquad (\epsilon^{\dot{A}\dot{B}}) : \mathcal{S}_- \to \mathcal{S}^-,$$

$$(\epsilon_{AB}) : \mathcal{S}^+ \to \mathcal{S}_+, \qquad (\epsilon_{\dot{A}\dot{B}}) : \mathcal{S}^- \to \mathcal{S}_-$$

is given, respectively, in terms of numerical indices by

$$\xi_B \mapsto \xi^A = \epsilon^{AB}\xi_B, \qquad \xi_{\dot{A}} \mapsto \xi^{\dot{B}} = \xi_{\dot{A}}\epsilon^{\dot{A}\dot{B}},$$

$$\xi^A \mapsto \xi_B = \xi^A\epsilon_{AB}, \qquad \xi^{\dot{B}} \mapsto \xi_{\dot{A}} = \epsilon_{\dot{A}\dot{B}}\xi^{\dot{B}},$$

where the anti-symmetric tensors are satisfying $\epsilon^{01} = \epsilon_{01} = -\epsilon^{\dot{0}\dot{1}} = -\epsilon_{\dot{0}\dot{1}} = 1$ (cf. [18]).

Let

$$\Gamma : (\mathcal{S}_+ \oplus \mathcal{S}_+) \otimes (\mathcal{S}_- \oplus \mathcal{S}_-) \to C^4$$

be defined by

$$\Gamma\left(f_{(iA)}, f_{\left(\begin{smallmatrix}\dot{B}\\ j\end{smallmatrix}\right)}\right) = \Gamma^{\mu}_{(iA)\left(\begin{smallmatrix}\dot{B}\\ j\end{smallmatrix}\right)} e_{\mu},$$

where $\{f_{(iA)}\}$ is the basis of $\mathcal{S}_+$, $\left\{f_{\left(\begin{smallmatrix}\dot{A}\\ i\end{smallmatrix}\right)}\right\}$ the basis of $\mathcal{S}_-$ and $\{e_\mu\}$ the basis of $C^4$ ($i = 1, 2$; $A = 0, 1$; $\dot{A} = \dot{0}, \dot{1}$; $\mu = 1, 2, 3, 4$). We denote $\Pi$ the parity change. When an element of $\Pi\mathcal{S}_+$, $\Pi\mathcal{S}_-$ and $C^4$ are written by $\theta^{iA} f_{(iA)}$, $\theta_i^{\dot{A}} f_{\left(\begin{smallmatrix}\dot{A}\\ i\end{smallmatrix}\right)}$ and $z^\mu e_\mu$, respectively, the coordinate of $H = C^4 \oplus \Pi\mathcal{S}_+ \oplus \Pi\mathcal{S}_+ \oplus \Pi\mathcal{S}_- \oplus \Pi\mathcal{S}_-$ is $(z^\mu, \theta^{iA}, \theta_i^{\dot{A}})$.

**Definition 2.2.** If the multiplication $\cdot$ on $H$ is defined by

$$\begin{pmatrix} z^\mu \\ \theta^{iA} \\ \theta_i^{\dot{A}} \end{pmatrix} \cdot \begin{pmatrix} w^\mu \\ \eta^{iA} \\ \eta_i^{\dot{A}} \end{pmatrix} = \begin{pmatrix} z^\mu + w^\mu + \Gamma^{\mu}_{(iA)\left(\begin{smallmatrix}\dot{B}\\ j\end{smallmatrix}\right)} \theta^{iA}\eta_j^{\dot{B}} + \Gamma^{\mu}_{(iB)\left(\begin{smallmatrix}\dot{A}\\ j\end{smallmatrix}\right)} \theta_j^{\dot{A}}\eta^{iB} \\ \theta^{iA} + \eta^{iA} \\ \theta_i^{\dot{A}} + \eta_i^{\dot{A}} \end{pmatrix},$$

then we call $(H, \cdot)$ the $N = 2$ supersymmetric Lie group.

**Proposition 2.1.** *Right invariant vector fields of the $N = 2$ supersymmetric Lie group $H$ is*

$$\Xi(H)^R = \left(\bigoplus_{\mu} C\frac{\partial}{\partial z^\mu}\right) \oplus \left(\bigoplus_{A,i} Cq_{iA}\right) \oplus \left(\bigoplus_{\dot{A},i} Cq^i_{\dot{A}}\right),$$

*where*

$$q_{iA} = \frac{\partial}{\partial\theta^{iA}} + \Gamma^{\mu}_{A\dot{B}}\theta_i^{\dot{B}}\frac{\partial}{\partial x^{\mu}} = \frac{\partial}{\partial\theta^{iA}} + \theta_i^{\dot{A}}\frac{\partial}{\partial x^{A\dot{A}}},$$
$$q^i_{\dot{A}} = \frac{\partial}{\partial\theta_i^{\dot{A}}} + \Gamma^{\mu}_{B\dot{A}}\theta^{i\dot{B}}\frac{\partial}{\partial x^{\mu}} = \frac{\partial}{\partial\theta_i^{\dot{A}}} + \theta^{iA}\frac{\partial}{\partial x^{A\dot{A}}},$$
$$\partial_{A\dot{A}} = \Gamma^{\mu}_{A\dot{A}}\partial_{\mu} = \frac{\partial}{\partial x^{A\dot{A}}}, \quad x^{A\dot{A}} = \Gamma_{\mu}^{A\dot{A}}x^{\mu}.$$

*And the* $N = 2$ *supersymmetric algebra is a super Lie algebra of graded dimension* $4|8$ *with four even generators* $\partial_\mu$ $(\mu = 1, \dots, 4)$ *and eight odd generators* $q_{iA}$, $q^i_{\dot{A}}$ $(i = 1, 2;\ A = 0, 1;\ \dot{A} = \dot{0}, \dot{1})$, *which form two* 2*-tuples of Weyl spinors of opposite type. The commutation relations are*

$$\{q_{iA}, q_{jB}\} = \{q^i_{\dot{A}}, q^j_{\dot{B}}\} = 0,$$
$$[\partial_\mu, \partial_\nu] = [\partial_\mu, q_{iA}] = [\partial_\mu, q^i_{\dot{A}}] = 0, \qquad \{q_{iA}, q^j_{\dot{A}}\} = 2\delta_i^j\Gamma^{\mu}_{A\dot{A}}\partial_\mu,$$

*where* $\Gamma^{\mu}_{A\dot{A}}$ *are the Pauli matrices.*

By a super vector bundle $E$ over a supermanifold $(M, \mathcal{O}(\wedge E))$ we mean a locally free $Z_2$-graded sheaf $E = E_0 + E_1$. We say that $E$ has rank $p|q$ if the local basis consist of $p$ even and $q$ odd sections. A super connection on $E$ is defined as usual by a linear mapping $\nabla^s : \Gamma(E) \to \Gamma(E \otimes T^*M)$ such that $f \in \mathcal{O}(\wedge E)$, $s \in \Gamma(E)$, $X, X_1, X_2 \in \Gamma(T^*M)$:

$$\nabla^s_X(fs) = (Xf)s + (-1)^{|X||f|} f\nabla^s_X s, \qquad \nabla^s_{X_1+X_2} = \nabla^s_{X_1} + \nabla^s_{X_2}, \qquad \nabla^s_{fX} = f\nabla^s_X.$$

The curvature of a super connection is the $\mathcal{O}(\wedge E)$-bilinear mapping:

$$F(X_1, X_2) = \nabla^s_{X_1}\nabla^s_{X_2} - (-1)^{|X_1||X_2|}\nabla^s_{X_2}\nabla^s_{X_1} - \nabla^s_{[X_1,X_2]}.$$

With respect to the basis $\{q_{iA}, q^i_A, \partial_{A\dot{A}}\}$ for the vector fields on $M = C^{4|4N}$ and to the local basis $\{s_1, s_2, \dots, s_p|s_{p+1}, \dots, s_{p+q}\}$ for $E$, we can give the super connections on $E$:

$$Q_{iA} = q_{iA} + \omega_{iA}, \qquad Q^i_{\dot{A}} = q^i_{\dot{A}} + \omega^i_{\dot{A}}, \qquad \nabla_{A\dot{A}} = \partial_{A\dot{A}} + A_{A\dot{A}}.$$

Then we can define the odd–odd curvature by

$$F_{iAjB} = \{Q_{iA}, Q_{jB}\}, \qquad F^{ij}_{\dot{A}\dot{B}} = \{Q^i_{\dot{A}}, Q^j_{\dot{B}}\}, \qquad F^j_{iA\dot{B}} = \{Q_{iA}, Q^j_{\dot{B}}\} - 2\delta_i^j\nabla_{A\dot{B}}.$$

The odd–even and even–even curvatures are

$$F_{iA,B\dot{B}} = [Q_{iA}, \nabla_{B\dot{B}}], \qquad F^i_{\dot{A},B\dot{B}} = [Q^i_{\dot{A}}, \nabla_{B\dot{B}}], \qquad F_{A\dot{A},B\dot{B}} = [\nabla_{A\dot{A}}, \nabla_{B\dot{B}}]$$

(cf. [8–10]).

**Theorem 2.1** (Harnad–Hurtubise–Legare–Shnider, 1985). *There is a one-to-one correspondence between*:

(1) *the complex super connection* $(A_{A\dot{A}}, \omega_{iA}, \omega^i_{\dot{A}})$ *subject to the constraints as follows*:

$$\{Q_{iA}, Q_{jB}\} + \{Q_{iB}, Q_{jA}\} = 0, \tag{1}$$

$$\{Q^i_{\dot{A}}, Q^j_{\dot{B}}\} + \{Q^i_{\dot{B}}, Q^j_{\dot{A}}\} = 0, \tag{2}$$

$$\{Q_{iA}, Q^j_{\dot{B}}\} - 2\delta^j_i \nabla_{A\dot{B}} = 0 \tag{3}$$

*and the following transverse gauge conditions*:

$$\theta^{iA}\omega_{iA} + \theta^i_{\dot{A}}\omega^{\dot{A}}_i = 0$$

*and*

(2) *the superfields* $(A_{A\dot{A}}, \chi_{iA}, \chi^i_{\dot{A}}, \phi_{ij})$ *subject to the N-supersymmetric Yang–Mills equations* ($N = 1, 2,$ *or* 3), *and implying the gauge condition, which are called the D-recursions*.

## 3. Super twistor space

We will define the $N$ super twistor space ($N = 1, 2, 3$ or 4) (cf. [7,15]).

**Definition 3.1.** The super twistor space $Z = Z^{3|N}$ is defined by the rank-$2|N$ vector bundle over the Riemann sphere $P^1$ obtained by tensoring the trivial bundle with the hyperplane bundle $\mathcal{O}_{P^1}(1)$ and the odd hyperplane bundle $\Pi\mathcal{O}_{P^1}(1)$:

$$Z^{3|N} = \mathcal{O}_{P^1}(1) \oplus \mathcal{O}_{P^1}(1) \oplus \underbrace{\Pi\mathcal{O}_{P^1}(1) \oplus \cdots \oplus \Pi\mathcal{O}_{P^1}(1)}_{N\text{-times}}.$$

The associated space is defined to be the space of sections of this bundle which is

$$M = \Gamma(P^1, Z) \simeq C^{4|4N}.$$

We will denote the holomorphically embedded in $P^1$ in $Z$ corresponding to the point $x = (x^{A\dot{A}}|\theta^{iA}, \theta^{\dot{A}}_i) \in M$ by $\hat{x}$. Introduce homogeneous coordinates $[\pi_{\dot{0}}, \pi_{\dot{1}}]$ on $P^1$ and homogeneous coordinates $([\omega^0, \omega^1]|c_i)$ ($i = 1, 2, 3$ or 4) on the fibers of $Z$. The general section of $Z$ is then given by

$$\omega^A = x^{A\dot{A}}\pi_{\dot{A}} + \theta^{iA}\theta^{\dot{A}}_i\pi_{\dot{A}}, \qquad c_i = \theta^{\dot{A}}_i\pi_{\dot{A}} \quad (i = 1, 2, 3, 4).$$

A point $z \in Z$ is represented by the subset of $M$ consisting of those $x \in M$ such that $\hat{x}$ contains $z$. This is the affine codimension $2|N$ hyperplane $\Sigma_{(z|\zeta)} = \Sigma^{2|3N}_{(z|\zeta)}$ in $M$, so called *self-dual super plane*, given by holding $([\omega^A, \pi_{\dot{A}}]|c_i)$ fixed in the above equation and letting the $x$-variables vary. We see that the tangent space of self-dual super plane is spanned by vectors

$$\left\{\pi^{\dot{A}}\frac{\partial}{\partial x^{A\dot{A}}}, \pi^{\dot{A}}\frac{\partial}{\partial\theta_i^{\dot{A}}}, \frac{\partial}{\partial\theta^{iA}}\right\}.$$

The correspondence space is

$$Y = \{((x|\theta), (z|\zeta)) \in M \times Z : z \in \hat{x}\}.$$

The restrictions to $Y$ of the projections on the two factors of $M \times Z$ define a double fibration

$$Z \overset{p_2}{\leftarrow} Y \overset{p_1}{\rightarrow} M,$$

where for each $(x|\theta)$ in $M$, $p_1^{-1}((x|\theta))$ is a copy of $\hat{x} = P^1$ while for each $(z|\zeta) \in Z$, $p_2^{-1}((z|\zeta))$ is $\Sigma_{(z|\zeta)}$, the space of $(x|\theta)$ such that $\hat{x}$ passes through $(z|\zeta)$. To understand these maps more explicitly, it will be convenient to introduce coordinates $(x^{A\dot{A}}|\theta^{iA}, \theta_i^{\dot{A}})$ on $M$ so that coordinates on $Y$ are

$$(x^{A\dot{A}}, [\pi_{\dot{A}}]|\theta^{iA}, \theta_i^{\dot{A}})$$

with

$$p_1((x^{A\dot{A}}, [\pi_{\dot{A}}]|\theta^{iA}, \theta_i^{\dot{A}})) = (x^{A\dot{A}}|\theta^{iA}, \theta_i^{\dot{A}})$$

and

$$p_2((x^{A\dot{A}}, [\pi_{\dot{A}}]|\theta^{iA}, \theta_i^{\dot{A}})) = ([x^{A\dot{A}}\pi_{\dot{A}} + \theta^{iA}\theta_i^{\dot{A}}\pi_{\dot{A}}, \pi^{\dot{A}}]|\theta_i^{\dot{A}}\pi_{\dot{A}}).$$

## 4. $N$=2 supersymmetric instantons

We refer to [5,6,14–16] for the definition of supersymmetric instantons. In this paper we adopt the definition of supersymmetric instantons as follows.

**Definition 4.1.** $N = 2$ supersymmetric instantons on $C^{4|8}$ is defined by the following conditions:

$$F^+_{A\dot{A},B\dot{B}} = 0, \qquad F_{iA,B\dot{B}} = 0, \qquad F_{iA,jB} = 0.$$

**Proposition 4.1.** *The following three conditions are equivalent*:

(1) $N = 2$ *supersymmetric instantons*.
(2) $F^+_{A\dot{A},B\dot{B}} = 0$, $\chi^i_{\dot{A}} = 0$, $\tilde{\phi} = 0$.
(3) $F^+_{A\dot{A},B\dot{B}} = 0$, $\epsilon^{AB}\nabla_{A\dot{A}}\chi^i_B = 0$, $\nabla_{A\dot{B}}\nabla^{A\dot{B}}\phi = \frac{1}{2}\epsilon_{ij}\{\chi^{iA}, \chi^j_A\}$.

**Proof.** The $N = 2$ constraints are equivalent to

$$F_{iA,jB} = 2\epsilon_{ij}\epsilon_{AB}\tilde{\phi}, \qquad F^{ij}_{\dot{A}\dot{B}} = 2\epsilon^{ij}\epsilon_{\dot{A}\dot{B}}\phi, \qquad F^j_{iA\dot{B}} = 0,$$

where $\tilde{\phi}$, $\phi$ are the bosonic scalar superfields, so called Higgs fields, defined by contraction of odd–odd indices in the super curvature. From the constraints (1), (2), (3) in Theorem 2.1 and the Bianchi identities we derive the following relations (cf. [9,10]):

$$\begin{aligned}
&Q_{iA}\tilde{\phi} = 0, \qquad Q^{i}_{\dot{A}}\tilde{\phi} = \epsilon^{ij}\chi_{j\dot{A}}, \qquad Q_{jA}\phi = \epsilon_{ij}\chi^{i}_{A}, \qquad Q^{i}_{\dot{A}}\phi = 0,\\
&F_{iA,B\dot{B}} = \epsilon_{AB}\chi_{i\dot{B}}, \qquad F^{i}_{\dot{A},B\dot{B}} = \epsilon_{\dot{A}\dot{B}}\chi^{i}_{B}, \qquad Q^{j}_{A}\chi_{i\dot{B}} = 2\delta^{j}_{i} f_{\dot{A}\dot{B}} + \epsilon_{\dot{A}\dot{B}} D^{j}_{i},\\
&Q_{iA}\chi^{j}_{B} = 2\delta^{j}_{i} f_{AB} - \epsilon_{AB} D^{j}_{i}, \qquad Q_{iA}\chi_{j\dot{A}} = 2\epsilon_{ij}\nabla_{A\dot{A}}\tilde{\phi}, \qquad Q^{i}_{\dot{A}}\chi^{j}_{A} = 2\epsilon^{ij}\nabla_{A\dot{A}}\phi,\\
&Q_{kA} D^{j}_{i} = -2\delta^{j}_{k}\epsilon^{\dot{A}\dot{B}}\nabla_{A\dot{A}}\chi_{i\dot{B}} + \delta^{j}_{i}\epsilon^{\dot{A}\dot{B}}\nabla_{A\dot{A}}\chi_{k\dot{B}} - 2\epsilon_{ki}[\chi^{j}_{A}, \tilde{\phi}],\\
&Q^{k}_{\dot{A}} D^{j}_{i} = 2\delta^{k}_{i}\epsilon^{AB}\nabla_{A\dot{A}}\chi^{j}_{B} - \delta^{j}_{i}\epsilon^{AB}\nabla_{B\dot{A}}\chi^{k}_{A} + 2\epsilon^{kj}[\chi_{iA}, \phi],\\
&F_{A\dot{A},B\dot{B}} = \epsilon_{AB} f_{\dot{A}\dot{B}} + \epsilon_{\dot{A}\dot{B}} f_{AB},
\end{aligned}$$

where $D^{j}_{i}$ is the auxiliary field, $f_{\dot{A},\dot{B}}$, $f_{AB}$ are, respectively, self-dual and anti-self-dual part of even–even curvature, fermions $(\chi^{1}_{A}, \chi_{1\dot{A}})$ and $(\chi^{2}_{A}, \chi_{2\dot{A}})$ are gaugino and Higgsino (cf. [8–10]).

$N = 2$ supersymmetric Yang–Mills field equations are

$$\begin{aligned}
&\epsilon^{AB}\nabla_{A\dot{B}} f_{CB} + \epsilon^{\dot{A}\dot{C}}\nabla_{C\dot{A}} f_{\dot{C}\dot{B}} + \{\chi^{i}_{C}, \chi_{i\dot{B}}\} + [\phi, \nabla_{C\dot{B}}\tilde{\phi}] + [\tilde{\phi}, \nabla_{C\dot{B}}\phi] = 0,\\
&\epsilon^{\dot{A}\dot{B}}\nabla_{A\dot{A}}\chi_{i\dot{B}} + \epsilon_{ij}[\chi^{j}_{A}, \tilde{\phi}] = 0, \qquad \epsilon^{AB}\nabla_{A\dot{A}}\chi^{i}_{B} + \epsilon^{ij}[\chi_{j\dot{A}}, \phi] = 0,\\
&\nabla_{A\dot{B}}\nabla^{A\dot{B}}\tilde{\phi} + [[\phi, \tilde{\phi}], \tilde{\phi}] - \tfrac{1}{2}\epsilon^{ij}\{\chi^{\dot{A}}_{i}, \chi_{j\dot{A}}\} = 0,\\
&\nabla_{A\dot{B}}\nabla^{A\dot{B}}\phi + [[\tilde{\phi}, \phi], \phi] - \tfrac{1}{2}\epsilon_{ij}\{\chi^{iA}, \chi^{j}_{A}\} = 0. \qquad \square
\end{aligned}$$

## 5. Supersymmetric Penrose–Ward correspondence

The following theorem is involved in the case of $N = 1$ [15]. Thus we will discuss only the case of $N = 2$.

**Theorem 5.1.** *Let $U \subset C^{4|8}$ be an open set such that the intersection of $U$ with every self-dual super plane $\Sigma$ that meets $U$ is connected and simply connected. Then there is a natural one-to-one correspondence between*:

(a) $N = 2$ *supersymmetric instantons on $U$ with structure group $GL(p|q, C)$ and*
(b) *holomorphic rank-$p|q$ super vector bundles $E$ over the $3|2$-dimensional super twistor space $Z$ such that $E$ restricted to $\hat{x}$ is trivial for all $(x|\theta) \in U$.*

**Proof.** We will describe how to go from (a) to (b) and then how to go from (b) to (a). Suppose that we are given an $N = 2$ supersymmetric instantons over $GL(p|q, C)$-super vector bundle $V$ defined on $U$. Note that the set of all $N = 2$ self-dual super plane $\Sigma = \Sigma_{(z|\zeta)}$ in $U$ is a $2|6$-dimensional super twistor space $Z$ of $U$. To construct a vector bundle $E$ over $Z$ we must assign a copy of the super vector space $C^{p|q}$ to each point $\tilde{\Sigma} = \tilde{\Sigma}^{2|6}$ of $Z$, this $C^{p|q}$

being the fiber $E_{\tilde{\Sigma}}$ over $\tilde{\Sigma}$. This assignment is given by the definition

$$E_{\tilde{\Sigma}} = \{\psi \in \Gamma(U \cap \Sigma, V) : \nabla|_{U\cap\Sigma}\psi = 0\}.$$

This definition requires it to be covariantly constant on $\Sigma$, i.e. to satisfy the propagation equation:

$$\nabla_{T^{A\dot{A}}}\psi = 0, \tag{4}$$

$$\nabla_{T_i^{\dot{A}}}\psi = 0, \tag{5}$$

$$\nabla_{T^{iA}}\psi = 0 \tag{6}$$

for all vector fields $T^{A\dot{A}}$, $T_i^{\dot{A}}$ and $T^{iA}$ tangent to $\Sigma$. Since $U \cap \Sigma$ is connected and simply connected, i.e. equivalent to the condition that the restriction of the super curvature $F$ to $U \cap \Sigma$ should vanish. Substituting $T^{A\dot{A}} = \lambda^A \pi^{\dot{A}} \partial_{A\dot{A}}$ for some spinor $\lambda^A \in \mathcal{S}^+$ and $T_i^{\dot{A}} = \alpha_i \pi^{\dot{A}}(\partial/\partial\theta_i^{\dot{A}})$, $T^{iA} = \beta^i(\partial/\partial\theta^{iA})$ for some odd constant $\alpha_i$, $\beta^i$ into (4)–(6), we have

$$\lambda^A \pi^{\dot{A}} \lambda^B \pi^{\dot{B}} F_{A\dot{A},B\dot{B}} = 0, \tag{7}$$

$$\alpha_i \pi^{\dot{A}} \lambda^B \pi^{\dot{B}} F^i_{\dot{A},B\dot{B}} = 0, \tag{8}$$

$$\beta^i \lambda^B \pi^{\dot{B}} F_{iA,B\dot{B}} = 0, \tag{9}$$

$$\alpha_i \alpha_j \pi^{\dot{A}} \pi^{\dot{B}} F^{ij}_{\dot{A}\dot{B}} = 0, \tag{10}$$

$$\beta^i \beta^j F_{iAjB} = 0, \tag{11}$$

$$\alpha_j \beta^i \pi^{\dot{B}} F^j_{iA\dot{B}} = 0. \tag{12}$$

Taking the inhomogeneous coordinate $[\pi^{\dot{0}}, \pi^{\dot{1}}] = [1, \zeta] \in P^1$, we will calculate from (7)–(10) and (12), respectively.

From (7), we see that

$$0 = \lambda^A \pi^{\dot{A}} \lambda^B \pi^{\dot{B}} F_{A\dot{A},B\dot{B}} = \lambda^A \lambda^B (F_{A\dot{0},B\dot{0}} + \zeta(F_{A\dot{1},B\dot{0}} + F_{A\dot{1},B\dot{0}}) + \zeta^2 F_{A\dot{1},B\dot{1}})$$

for any $\lambda^A$, $\lambda^B$ and $\zeta$. Hence

$$F_{A\dot{0},B\dot{0}} = F_{A\dot{1},B\dot{1}} = 0, \qquad F_{A\dot{0},B\dot{1}} = F_{A\dot{1},B\dot{0}} = 0$$

for any $A$ and $B$. This is the anti-self-dual condition.

From (8), we see that

$$0 = \alpha_i \pi^{\dot{A}} \lambda^B \pi^{\dot{B}} F_{i\dot{A},B\dot{B}} = \alpha_i \lambda^B (F_{i\dot{0},B\dot{0}} + \zeta(F_{i\dot{0},B\dot{1}} + F_{i\dot{1},B\dot{0}}) + \zeta^2 F_{i\dot{1},B\dot{1}})$$

for any $\alpha_i$, $\lambda^B$ and $\zeta$. Hence

$$F_{i\dot{0},B\dot{0}} = F_{i\dot{1},B\dot{1}} = 0, \tag{13}$$

$$F_{i\dot{0},B\dot{1}} = F_{i\dot{1},B\dot{0}} = 0 \tag{14}$$

for any $B$. From a Bianchi identity $F_{i\dot{A},B\dot{B}} = \epsilon_{\dot{A}\dot{B}}\chi_{iB}$, we obtain that the conditions (13) and (14) are automatically satisfied.

From (9), we obtain the essential property that

$$F_{iA,B\dot{B}} = 0.$$

This condition is equivalent to $\chi_{i\dot{A}} = 0$.

From (10), we see in the same way that

$$F^{ij}_{\dot{0}\dot{0}} = F^{ij}_{\dot{1}\dot{1}} = 0, \qquad F^{ij}_{\dot{0}\dot{1}} = F^{ij}_{\dot{1}\dot{0}} = 0.$$

This condition is equivalent to constraint (2).

From (11), we have $F_{iAjB} = 0$. This condition is equivalent to $\tilde{\phi} = 0$. Note that this is the condition stronger than the constraint (1). We also see that (12) is equivalent to constraints (3). Therefore (4)–(6) are equivalent to the super instantons fields.

Let us move on now to showing how to go from (b) to (a). Let $E$ be a holomorphic rank-$p|q$ super vector bundle over the super twistor space $Z$, such that $E|\hat{x}$ is trivial for all $(x|\theta) \in U$. We have to construct a gauge potential $A$ on $U$. We define a bundle $V \to U$ by

$$V_{(x|\theta)} = \Gamma(\hat{x}, E|\hat{x}).$$

By hypothesis, the restriction of $E$ to $\hat{x}$ is the product bundle $\hat{x} \times C^{p|q}$. In this trivialization, global sections of $E|\hat{x}$ are holomorphic maps $\hat{x} \to C^{p|q}$. By Liouville's theorem they are constant, and hence $V_{(x|\theta)}$ is a vector space of dimension $p|q$. Therefore we have a rank-$p|q$ vector bundle $V$ over $U$. We now want to define a super connection $(A, \omega)$ on $V$, which means knowing how to parallel propagate vectors $\psi$ in $E$ along curves in $U$. This in turn is equivalent to knowing how to propagate $\psi$ in null directions, since the null vectors span the whole tangent space of $U$. Any null vector is tangent to a unique self-dual super plane $\Sigma$, and the various points on $\Sigma$ correspond to lines in $Z$ passing through $\tilde{\Sigma}$. If $\hat{x}$ and $\hat{y}$ are two lines intersecting at $\tilde{\Sigma}$, then $\Gamma(E|\hat{x}) \simeq \Gamma(E|\hat{y})$. So we know how to propagate a vector $\psi$ from $x$ to $y$, and hence we know the super connection on $V$. By construction, this connection is flat, i.e. $\nabla_{U\cap\Sigma}\psi = 0$, on all self-dual super planes, and therefore is a supersymmetric instanton. □

Actually, the above argument is incomplete, since we should check that the propagation rule really does define a connection. We will give an explicit description of how to extract the super connection from the transition functions $g((Z^{\alpha}|c_i))$ for the super vector bundle $E$ in Appendix A. But remains of this proof is of analogy for argument of Ward and Wells [18].

We will restrict the complex super space $C^{4|8}$ to the Euclidean super space $R^{4|8}$.

**Theorem 5.2.** *There is a natural one-to-one correspondence between*:

(a) $N = 2$ *supersymmetric instantons on* $R^{4|8}$ *with structure group* $GL(n, C)$ *and*
(b) *holomorphic rank-n vector bundles E over the* 3|2-*dimensional super twistor space Z such that E restricted to* $\hat{x}$ *is trivial for all* $(x|\theta) \in R^{4|8}$.

**Proof.** Let $P^{3|2}$ be a 3|2-dimensional complex projective super space. Let $\sigma : P^{3|2} \to P^{3|2}$ denote the anti-holomorphic involution, so called *real structure*, defined by

$$\sigma(\omega^0, \omega^1, \pi_{\dot{0}}, \pi_{\dot{1}}|\zeta^1, \zeta^2) = (\overline{\omega^1}, -\overline{\omega^0}, \overline{\pi_{\dot{i}}}, -\overline{\pi_{\dot{0}}}|\overline{\zeta^2}, -\overline{\zeta^1}).$$

For any point $(z|\zeta) \in P^{3|2}$, the line $\hat{x}$ joining $(z|\zeta)$ to $\sigma((z|\zeta))$ is a real line, i.e. $\hat{x}$ is invariant under $\sigma$.

The real structure on $C^{4|8}$ is defined by (cf. [10])

$$\overline{x^{B\dot{B}}} = \epsilon^{AB} x^{A\dot{A}} \epsilon^{\dot{A}\dot{B}}, \qquad \overline{\theta^{iA}} = \epsilon^{AB} \epsilon^{i}_{j} \theta^{jB}, \qquad \overline{\theta^{\dot{A}}_{i}} = \epsilon^{\dot{A}\dot{B}} \epsilon^{j}_{i} \theta^{\dot{B}}_{j},$$

where the anti-symmetric tensors $\epsilon^{AB}$, $\epsilon_{AB}$, $\epsilon^{\dot{A}\dot{B}}$ and $\epsilon_{\dot{A}\dot{B}}$ ($\epsilon^{01} = \epsilon_{01} = -\epsilon^{\dot{0}\dot{1}} = -\epsilon_{\dot{0}\dot{1}} = 1$) will be used for raising and lowering of spinor indices:

$$\xi^A = \epsilon^{AB}\xi_B, \qquad \xi^{\dot{B}} = \xi_{\dot{A}}\epsilon^{\dot{A}\dot{B}}, \qquad \xi_B = \xi_A\epsilon_{AB}, \qquad \xi_{\dot{A}} = \epsilon_{\dot{A}\dot{B}}\xi^{\dot{B}}$$

and $\epsilon^{i}_{j} = \epsilon^{AB}$.

Next, now given a complex $n$-vector bundle $V$ over $R^{4|8}$, with super connection $\nabla^s$, we can pull both $V$ and $\nabla^s$ back to $Z$, thereby obtaining a vector bundle $E$ over $Z$, with super connection $\tilde{\nabla}^s$. At this stage, $E$ is only a differentiable bundle. We want to endow $E$ with a super holomorphic structure. But using demonstrated in Atiyah et al. [3], we see in the same way that $E$ has a super holomorphic structure [13]. Note that the double fibration deduce to

$$Z^{3|2} \overset{p_2}{\leftarrow} R^{4|8} \times P^1 \overset{p_1}{\rightarrow} R^{4|8}$$

for each $(z|\zeta)$ in $Z$, $p_2^{-1}((z|\zeta))$ become the real 0|4-real-dimensional super subspace in $\Sigma^{2|6}$ spanned by $\{\partial/\partial\theta^{A}_{i}\}$. □

Restricting the structure group $GL(n, C)$ to $SU(n)$, we also obtain the following.

**Theorem 5.3.** *There is a natural one-to-one correspondence between*:

(a) $N = 2$ *supersymmetric instantons on* $R^{4|8}$ *with structure group* $SU(n)$ *and*
(b) *holomorphic rank-n vector bundles E over the* 3|2-*dimensional super twistor space Z such that*
   (i) $E|\hat{x}$ *is trivial for all* $(x|\theta) \in R^{4|8}$;
   (ii) det $E$ *is trivial*;
   (iii) $E$ *admits a positive real form.*

**Proof.** To reduce the structure group (gauge group) to $SU(n)$, we equip each fiber of $V$ with a unitary structure. The unitary structure on $V$ can be given by an anti-linear isomorphism $\tilde{\tau} : V \to V^*$ such that $\langle\psi, \tilde{\tau}\varphi\rangle$ is a positive Hermitian form ($V^*$ denotes the dual of $V$). Here $\langle,\rangle$ denotes the natural pairing between $V$ and $V^*$. Passing to $E$ we use $\tilde{\tau}$ to define a lifting $\tau$ of the real structure $\sigma$ on $Z$, namely we define a commutative diagram:

$$\begin{array}{ccc} \underline{E} & \xrightarrow{\ \tau\ } & \underline{E}^* \\ \downarrow & & \downarrow \\ \underline{Z} & \xrightarrow{\ \sigma\ } & \underline{Z} \end{array}$$

where $\tau$ maps $E_{(z|\zeta)}$ to $E^*_{\sigma((z|\zeta))}$, and

$$\langle \xi, \tau\eta \rangle = \overline{\langle \eta, \tau\xi \rangle}, \quad \xi \in E_{(z|\zeta)}, \quad \eta \in E_{\sigma((z|\xi))},$$

which is called a *positive real form* on $E$.

Conversely, having such a structure $\tau$ on $Z$ guarantees that the corresponding gauge field on $R^{4|8}$ admits an Hermitian structure, i.e. it is a $U(n)$-gauge field. Assuming that $E|\hat{x}$ is trivial for all $(x|\theta) \in R^{4|8}$, then $\tau$ induces a nondegenerate Hermitian form $\tilde{\tau}$ on the space $V_{(x|\theta)} = \Gamma(\hat{x}, E|\hat{x})$ of holomorphic sections of $E|\hat{x}$ for each $(x|\theta) \in R^{4|8}$. □

## 6. ADHM-matrix representations

We next describe Horrock's monad construction of holomorphic rank-2 super vector bundles on 3|2-dimensional complex projective super space $P^{3|2}$. The monad construction may be described in terms of the following data.

*Data*. The super linear algebra for the monad construction of holomorphic super vector bundles $E \to P^{3|2}$ corresponding to $N = 2$ supersymmetric instantons on a $SU(2)$-bundle $V \to R^{4|8}$ with $c_2(V) = k$ is given by the following:

(i) A map $\sigma : C^{4|2} \to C^{4|2}$ defined by

$$\sigma(z|\zeta) = \sigma(z^1, z^2, z^3, z^4|\zeta^1, \zeta^2) = (\bar{z}^2, -\bar{z}^1, \bar{z}^4, -\bar{z}^3|\bar{\zeta}^2, -\bar{\zeta}^1).$$

(ii) A complex super vector space $W$, with $\dim_C W = k|2$ and a conjugate linear map $\sigma_W : W \to W$, $\sigma_W^2 = 1$.

(iii) A complex super vector space $V$ (note that we use the same symbol $V$ as the vector bundle over $R^{4|8}$), with $\dim_C V = (2k+2)|4$, with a symplectic form $b$ and conjugate linear map $\sigma_V : V \to V$, so that $\sigma_V^2 = -1$, and satisfying:

(a) The form $b$ is compatible with $\sigma_V$, in the sense that

$$b(\sigma_V u, \sigma_V v) = \overline{b(u, v)}.$$

(b) The induced Hermitian form $h(u, v) = b(u, \sigma_V u)$ is required to be positive definite.

(iv) A super linear map $A((z|\zeta)) : W \to V$, depending linearly on $(z|\zeta) = (z^1, z^2, z^3, z^4|\zeta^1, \zeta^2)$, so

$$A((z|\zeta)) = \sum_{\alpha=1}^{4} A_\alpha z^\alpha + \sum_{i=1}^{2} B_i \zeta^i,$$

where $A_\alpha$ and $B_i$ are $(2k+2|4)\times(k|2)$-constant even and odd matrices satisfying:

(a) Rank condition: for all $(z|\zeta)\neq(0|0)$, $\dim_C \mathrm{Im}\, A((z|\zeta))=k$.
(b) Isotropic condition: for all $(z|\zeta)\neq(0|0)$, $\mathrm{Im}\, A((z|\zeta))$ is an isotropic subspace of $V$, so that $\mathrm{Im}\, A((z|\zeta))\subset(\mathrm{Im}\, A((z|\zeta)))^\circ$, where $(\mathrm{Im}\, A((z|\zeta)))^\circ=\{v\in V: b(u,v)=0, u\in \mathrm{Im}\, A((z|\zeta))\}$.
(c) Reality condition: for all $(z|\zeta)\in C^{4|2}$, $w\in W$, then $\sigma_V\{A((z|\zeta))w\}=A(\sigma(z|\zeta))\sigma_W w$.

The symplectic form $b: V\otimes V\to C$ induces an isomorphism $b: V\to V^*$ given by $v\mapsto b(v)=b(\cdot,v)$. Since $A^*((z|\zeta)): V^*\to W^*$, we obtain a map $A^*((z|\zeta))b: V\to W^*$ defined by

$$(A^*((z|\zeta))b(v))w=b(A((z|\zeta))w,v),\quad u\in V,\quad w\in W.$$

We then have the corresponding monad:

$$0\to W\otimes\mathcal{O}_{P^{3|2}}(-1)\overset{A}{\to}\underline{V}\overset{A^*b}{\to}W\otimes\mathcal{O}_{P^{3|2}}(1)\to 0,$$

where $\underline{V}$ and $\mathcal{O}_{P^{3|2}}(-1)$ denote the trivial bundle $V\times P^{3|2}\to P^{3|2}$ and the tautological line bundle on $P^{3|2}$, respectively. Then the bundle $E\to P^{3|2}$ is defined as $\mathrm{Ker}(A^*b)/\mathrm{Im}\, A$, with fibers

$$E_{(z|\zeta)}=\frac{\mathrm{Ker}(A^*((z|\zeta))b)}{\mathrm{Im}\, A((z|\zeta))}=\frac{(\mathrm{Im}\, A((z|\zeta)))^\circ}{\mathrm{Im}\, A((z|\zeta))}$$

for $(z|\zeta)\in P^{3|2}$. If we recall that $V\simeq C^{2k+2|4}$, $W\simeq C^{k|2}$, then we see that the above monad is equivalent to

$$\begin{aligned}&0\to\mathcal{O}^k_{P^{3/2}}(-1)\oplus(\Pi\mathcal{O}_{P^{3|2}}(-1))^2\\&\overset{A}{\to}\mathcal{O}^{2k+2}_{P^{3|2}}\oplus(\Pi\mathcal{O}_{P^{3|2}})^4\overset{A^*b}{\to}\mathcal{O}^k_{P^{3|2}}(1)\oplus(\Pi\mathcal{O}_{P^{3|2}}(1))^2\to 0.\end{aligned}$$

We have $C^{4|4}\otimes_C W\simeq H^{2|2}\otimes_R W_R$, and the induced map $\sigma\otimes\sigma_W$ on $C^{4|4}\otimes_C W$ corresponds to left multiplication by $j$ on the left quaternion super vector space $H^{2|2}\otimes_R W_R$. The complex super linear map $A: C^{4|4}\otimes_C W\to V$ may now be viewed as a map

$$A: H^{2|2}\otimes_R W_R\to V$$

and compatibility of $A((z|\zeta))$ with $\sigma\otimes\sigma_W$ is equivalent to requiring that $A$ be quaternion super linear. If we take a real basis of $W_R$ and an orthogonal $H$-basis of $V$, so that $V$ gets identified with $H^{k+1|2}$, then $A$ is described by four matrices $C$, $D$, $G$ and $H$. The column-vectors of $C$ are the image under $A$ of $(1,0|0,0)\otimes\{\text{basis vectors of } \mathrm{W_R}\}$ and $D$, $G$ or $H$ are similarly defined replacing $(1,0|0,0)$ by $(0,1|0,0)$, $(0,0|\xi^1,0)$, $(0,0|0,\xi^2)$ in $H^{2|2}$, respectively. We use matrices as right multipliers here since our scalars act on the left, so that $C$, $D$, $G$ and $H$ are $(k+1|2)\times(k|2)$-matrices. Regarded as a matrix function of the coordinate of $(x, y) \in H^{2|0}$ and $(\theta^{iA}, \theta_i^{\dot{A}}) \in H^{0|2} \simeq R^{0|8}$ we then have

$$A(x, y|\theta^{iA}, \theta_i^{\dot{A}}) = xC + yD + \sum_{i=1}^{2} G_{iA}\theta^{iA} + \sum_{i=1}^{2} H^i_{\dot{A}}\theta_i^{\dot{A}},$$

where $C$ and $D$ are $(k+1|2)\times(k|2)$-even matrices and $G_{iA}$ and $H^i_{\dot{A}}$ are $(k+1|2)\times(k|2)$-odd matrices.

Rank condition is equivalent to $A(x, y|\theta^{iA}, \theta_i^{\dot{A}})$ has maximal rank for all $(x, y|\theta^{iA}, \theta_i^{\dot{A}}) \neq (0, 0|0, 0)$. The columns of $A(x, y, |\theta^{iA}, \theta_i^{A})$ then span a subspace of $H^{k+1|2}$ having dimension $(k|2)$ and depending only on the ratio $xy^{-1}$, i.e. on the point of quaternionic super projective space $HP^{1|2}$. The orthogonal complement is then a subspace $V_{(x,y|\theta^{iA},\theta_i^{\dot{A}})}$ of quaternion super dimension $1|0$. The $Sp(1)$-bundle $V \to HP^{1|2}$ is obtained by setting

$$V_{(x,y|\theta^{iA},\theta_i^{\dot{A}})} = (\mathrm{Im}\, A(x, y|\theta^{iA}, \theta_i^{\dot{A}}))^{\perp} \subset H^{k+1|2}, \quad ([x, y]|\theta^{iA}, \theta_i^{\dot{A}}) \in HP^{1|2}.$$

We now have the quaternionic bundle exact sequence over $HP^{1|2}$ given by

$$0 \to V \to H^{k+1|2} \to kL \oplus \Pi L \oplus \Pi L \to 0,$$

where $L \to HP^{1|2}$ denotes the tautological quaternionic line bundle and $kL = L \oplus \cdots \oplus L$. If we restrict to $R^{4|8} \subset HP^{1|2}$ then we can take affine coordinate $(x, 1|\theta^{iA}, \theta_i^{\dot{A}})$ and $A(x|\theta^{iA}, \theta_i^{\dot{A}}) = A(x, 1|\theta^{iA}, \theta_i^{\dot{A}})$. Putting $A_{(x|\theta)} = A(x|\theta^{iA}, \theta_i^{\dot{A}})$, we obtain the following.

**Theorem 6.1.** *Quaternionic super matrices $A_{(x|\theta)}$ satisfying*

(i) *$A_{(x|\theta)}$ has rank k for all $(x|\theta) \neq (0|0)$,*
(ii) *$A^{\dagger}_{(x|\theta)}A_{(x|\theta)}$ is real, and*
(iii) *$G_{iA} = 0$ for all i, A*

*give rise to a supersymmetric instanton on a $Sp(1)$-bundle V over $R^{4|8}$ with Chern number k.*

**Proof.**

$$0 \to H^{1|0} \overset{B_{(x|\theta)}}{\to} H^{k+1|2} \overset{A^*_{(x|\theta)}}{\to} H^{k|2} \to 0.$$

Choosing a gauge for the bundle $V$ will give rise to linear maps $B_{(x|\theta)} : H^{1|0} \to H^{k+1|2}$ whose image is just $V_{(x|\theta)} \subset H^{k+1|2}$. If inner products are fixed so that $B_{(x|\theta)}$ is an orthogonal gauge then orthogonal projection $P_{(x|\theta)}$ onto $V_{(x|\theta)}$ is given by $P_{(x|\theta)} = B_{(x|\theta)}B^{\dagger}_{(x|\theta)}$, while $B^{\dagger}_{(x|\theta)}B_{(x|\theta)} = 1$. To compute the super covariant derivative $\nabla$ in the gauge $B_{(x|\theta)}$ we put $f = B_{(x|\theta)}g$ where $g$ is now a function on $R^{4|8}$ with values in $H^{1|0}$ and find

$$\nabla(B_{(x|\theta)}g) = P_{(x|\theta)}d(B_{(x|\theta)}) = B_{(x|\theta)}\{dg + B^{\dagger}_{(x|\theta)}(dB_{(x|\theta)})g\}$$

showing that the super connection form $\mathcal{A}$ is given by

$$\mathcal{A} = B^{\dagger}_{(x|\theta)} dB_{(x|\theta)},$$

when $d$ is flat super connection on $\underline{H}^{k+1|2}$. The super curvature $F_{\mathcal{A}}$ corresponding to the super connection $\mathcal{A}$ is given by

$$F_{\mathcal{A}} = P_{(x|\theta)} dA_{(x|\theta)} \rho^{-2} dA^{\dagger}_{(x|\theta)} P_{(x|\theta)}, \tag{15}$$

where $\rho^2 = A^{\dagger}_{(x|\theta)} A_{(x|\theta)}$. Substituting for $A_{(x|\theta)}$ in (15) gives the following expression for the super curvature $F_{\mathcal{A}}$:

$$\begin{aligned} F_{\mathcal{A}} = {} & P_{(x|\theta)}\{C dx \rho^{-2} d\bar{x} C^{\dagger} + C dx \rho^{-2} (\Sigma d\overline{\theta^{iA}} G^{\dagger}) + C dx \rho^{-2} (\Sigma d\overline{\theta^{\dot{A}}_{i}} H^{\dagger}) \\ & + (G \Sigma d\theta^{iA}) \rho^{-2} d\bar{x} C^{\dagger} + (\Sigma G d\theta^{iA}) \rho^{-2} (\Sigma d\overline{\theta^{iA}} G^{\dagger}) \\ & + (\Sigma G d\theta^{iA}) \rho^{-2} (\Sigma d\overline{\theta^{\dot{A}}_{i}} H^{\dagger}) + (\Sigma H d\theta^{\dot{A}}_{i}) \rho^{-2} d\bar{x} C^{\dagger} \\ & + (\Sigma H d\theta^{\dot{A}}_{i}) \rho^{-2} (\Sigma d\overline{\theta^{iA}} G^{\dagger}) + (\Sigma H d\theta^{\dot{A}}_{i}) \rho^{-2} (\Sigma d\overline{\theta^{\dot{A}}_{i}} H^{\dagger})\} P_{(x|\theta)}. \end{aligned}$$

From Definition 4.1 of the supersymmetric instantons, we see that $\rho^{-2}$ is real and $G_{iA} = 0$ for all $i$, $A$. □

## Appendix A

Giving an explicit description of how to extract the super connection from the transition function for the super vector bundle $E$, we will complete the proof of Theorem 5.1.

**Proof of Theorem 5.1.** If $\{W_0, W_1, \dots\}$ is an open covering of $Z$, for which $E|W_j$ is trivial, then there are transition functions $g_{ij}$ which are super matrix-valued functions of the form

$$g_{ij} : W_i \cap W_j \to GL(p|q; C).$$

The super twistor space $Z$ can be covered by two coordinate charts $W$ and $\underline{W}$ defined by

$$W = \{([\omega^A, \pi_{\dot{A}}]|c_i) : \pi_{\dot{1}} \neq 0\}$$

and

$$\underline{W} = \{([\omega^A, \pi_{\dot{A}}]|c_i) : \pi_{\dot{0}} \neq 0\}.$$

The super vector bundle $E$ is specified by giving a holomorphic $(p|q) \times (p|q)$ transition matrix $g((z^{\alpha}|c_i))$ on the intersection $W \cap \underline{W}$. The transition relation is

$$\underline{\xi} = g((z^{\alpha}|c_i))\xi,$$

where $\xi$ and $\underline{\xi}$ are column $p|q$-vectors whose components serve as coordinate on the fibers of $E$ above $W$ and $\underline{W}$, respectively. The first step is to restrict $E$ to a line $\hat{x}$ in $Z$. This is achieved by substituting $\omega^A = x^{AA}\pi_{\dot{A}} + \theta^{iA}\theta_i^{\dot{A}}\pi_{\dot{A}}$ and $c_i = \theta_i^{\dot{A}}\pi_{\dot{A}}$ into $g((z^\alpha|c_i))$, hence obtaining the transition matrix

$$G = G(x, \pi_{\dot{A}}, \theta) = g([x^{A\dot{A}}\pi_{\dot{A}} + \theta^{iA}\theta_i^{\dot{A}}\pi_{\dot{A}}, \pi_{\dot{A}}]|\theta_i^{\dot{A}}\pi_{\dot{A}})$$

for the bundle $E|\hat{x}$ over $\hat{x}$. We must now find the holomorphic sections of $E|\hat{x}$, and this can be done as follows. Find nonsingular $(p|q) \times (p|q)$ matrices $H = H(x, \pi_{\dot{A}}, \theta)$ and $\underline{H} = \underline{H}(x, \pi_{\dot{A}}, \theta)$ with $H$-holomorphic for all $\pi_{\dot{A}} \in W \cap \hat{x}$ and $\underline{H}$-holomorphic for all $\pi_{\dot{A}} \in \underline{W} \cap \hat{x}$, such that the *Birkhoff splitting formula*

$$G = \underline{H}H^{-1} \tag{A.1}$$

is valid on $W \cap \underline{W} \cap \hat{x}$. Since $E|\hat{x}$ is trivial, such matrices $H$ and $\underline{H}$ must exist. Each section of $E|\hat{x}$ is given by

$$\xi = H\psi, \qquad \underline{\xi} = \underline{H}\psi,$$

where $\psi$ is a constant $p|q$-vector with respect to $\pi_{\dot{A}}$. The negative odd spinor connection $\omega^i_{\dot{A}}$ is obtained by differentiation along a null vector $T_i^{\dot{A}} = \alpha_i\pi^{\dot{A}}(\partial/\partial\theta_i^{\dot{A}})$ for some odd constant $\alpha_i$. This gives

$$\begin{aligned} 0 &= \nabla_{T_i^{\dot{A}}}\psi = \alpha_i\pi^{\dot{A}}(\partial_{i\dot{A}}\psi + \omega^i_{\dot{A}}\psi) = \alpha_i\pi^{\dot{A}}(\partial_{i\dot{A}}(H^{-1}\xi) + \omega^i_{\dot{A}}\psi) \\ &= \alpha_i\pi^{\dot{A}}((\partial_{i\dot{A}}H^{-1})\xi + \omega^i_{\dot{A}}\psi) = \alpha_i\pi^{\dot{A}}(-H^{-1}(\partial_{i\dot{A}}H)H^{-1}\xi + \omega^i_{\dot{A}}\psi) \\ &= \alpha_i\pi^{\dot{A}}(-H^{-1}(\partial_{i\dot{A}}H) + \omega^i_{\dot{A}})\psi, \end{aligned}$$

which holds for all $\psi$ and for all $\alpha_i$. So we deduce that $\omega^i_{\dot{A}}$ is given

$$\pi^{\dot{A}}\omega^i_{\dot{A}} = H^{-1}(\pi^{\dot{A}}\partial_{i\dot{A}}H). \tag{A.2}$$

In the same way, the positive odd spinor connection $\omega_{iA}$ is obtained by differentiating along a null vector $T^{iA} = \beta^i(\partial/\partial\theta^{iA})$ for some odd constant $\beta^i$. This gives

$$\omega_{iA} = H^{-1}(\partial_{iA}H). \tag{A.3}$$

The even connection $A_{A\dot{A}}$ is also obtained by differentiating along a null vector $T^{A\dot{A}} = \lambda^A\pi^{\dot{A}}\partial_{A\dot{A}}$ for some even spinor $\lambda^A$ (cf. [18]):

$$\pi^{\dot{A}}A_{A\dot{A}} = H^{-1}(\pi^{\dot{A}}\partial_{A\dot{A}}H). \tag{A.4}$$

For the super connections $(A_{A\dot{A}}, \omega_{iA}, \omega^i_{\dot{A}})$ to be well-defined, we must prove that the right-hand side of (A.2) and (A.4) are linear in $\pi^{\dot{A}}$, respectively. To prove it, operate on the splitting formula $G = \underline{H}H^{-1}$ with $\pi^{\dot{A}}\partial_{i\dot{A}}$ and $\pi^{\dot{A}}\partial_{A\dot{A}}$, respectively (cf. [18, (8.1.6)]). Then we have

$$H^{-1}(\pi^{\dot{A}}\partial_{i\dot{A}}H) = \underline{H}^{-1}(\pi^{\dot{A}}\partial_{i\dot{A}}\underline{H}), \tag{A.5}$$

$$H^{-1}(\pi^{\dot{A}}\partial_{A\dot{A}}H) = \underline{H}^{-1}(\pi^{\dot{A}}\partial_{A\dot{A}}\underline{H}). \tag{A.6}$$

Now the left-hand side of (A.5) and (A.6) are holomorphic for $\pi_{\dot{A}} \in W \cap \hat{x}$, while the right-hand side are holomorphic for $\pi_{\dot{A}} \in \underline{W} \cap \hat{x}$. Thus both sides are holomorphic on the whole Riemann sphere $\hat{x}$, and in addition are homogeneous of degree 1 in $\pi_{\dot{A}}$. So both sides must be linear in $\pi_{\dot{A}}$, which was what we wanted to prove. □

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